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Note

Adjacency on the constrained assignment problem

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Abstract

Let $Q_{c,r}$ be the integer hull of the intersection of the assignment polytope with a given hyperplane $H = \{x = (x_{ij}) \in \mathbb{R}^{n \times n} : \sum_{i=1}^n \sum_{j=1}^n c_{ij}x_{ij} = r\}$. We show that the problem of checking whether two given extreme points of $Q_{c,r}$ are nonadjacent on $Q_{c,r}$ is solvable in $\mathcal{O}(n^5)$ time if $c = (c_{ij})$ is a 0–1 matrix, and that it is NP-Complete if c is a general integer matrix. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

When a convex polytope K is specified through a system of linear constraints, checking whether two given extreme points of it are adjacent involves computing the rank of a set of vectors, which can be carried out very efficiently. This result is implicitly used in the execution of the simplex algorithm of linear programming.

Many combinatorial optimization problems can be modeled as 0–1 integer programs. Let P denote the set of feasible solutions of the *linear programming relaxation* of such a problem, and P_I the convex hull of integer (i.e. 0–1) points of P . We are given a linear constraint representation for P , but usually not for P_I . Without explicitly using a full linear constraint representation, efficient variants of the simplex method have been developed for some combinatorial optimization problems; for example, matching and edge covering problems, minimum cost spanning tree problems, etc., and in

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all such cases, checking adjacency of two extreme points on the corresponding polytope P_I turns out to be polynomially solvable (for example, see [6] for an efficient criterion to check adjacency on the matching polytope). In the same publication, Chvátal has shown that whereas the maximum stable set problem is NP-complete, adjacency checking on the stable set polytope is polynomially solvable. Papadimitriou [16] and Chung [5] established that checking nonadjacency of extreme points in the polytopes P_I associated with NP-hard combinatorial optimization problems such as the traveling salesman problem, the set covering problem, the 0–1 knapsack problem, the maximum cost simple chain problem are all NP-complete; see also [9, 10, 18]. These results seem to suggest that a fundamental requirement for a combinatorial optimization problem to be efficiently solvable is that checking adjacency of extreme points on the polytope P_I associated with it be polynomial, even though the converse may not be true. Hence when faced with a combinatorial optimization problem of unknown complexity, studying the complexity of adjacency checking on the polytope P_I associated with it may shed some light.

In this paper we focus attention on the usual assignment problem on the complete $n \times n$ bipartite graph $K_{n,n}$ with an additional equality constraint, an important problem in core management of nuclear reactors [3, 8, 12]. The problem is

$$\begin{aligned}
 \min \quad & \sum_i \sum_j d_{ij} x_{ij} \\
 \text{subject to} \quad & \sum_j x_{ij} = 1, \quad i = 1 \text{ to } n, \\
 & \sum_i x_{ij} = 1, \quad j = 1 \text{ to } n-1, \\
 & x_{ij} \in \{0, 1\} \quad i, j = 1 \text{ to } n, \\
 & \sum_i \sum_j c_{ij} x_{ij} = r.
 \end{aligned} \tag{1}$$

Let $Q_{c,r}$ denote the convex hull of the set of feasible solutions of (1). When $c = (c_{ij})$ is a general integer matrix, even finding a feasible solution of (1) is NP-hard [4], but the complexity of (1) when c is 0–1 is unknown; see [1, 11, 14, 17] for related results. To shed some light on the complexity of (1), particularly when c is 0–1, we investigate the complexity of checking adjacency of two extreme points of $Q_{c,r}$. Our results show that when c is a 0–1 matrix there is an efficient adjacency routine for (1), but it is not clear whether there is an efficient algorithm for (1) in this special case.

2. Main results

Let $K_A = \{x = (x_{ij}) : x \geq 0 \text{ satisfies the first two constraints in (1)}\}$ = the assignment polytope of order n . We will call feasible solutions of (1), *feasible assignments*, these are extreme points of $Q_{c,r}$. For each feasible assignment x^k , let $M^k = \{(i, j) \in I_1 \times I_2 : I_1 = I_2 = \{1, \dots, n\}, x_{ij}^k = 1\}$ denote the corresponding *feasible perfect matching* in $K_{n,n}$.

Given two distinct extreme points, x^1, x^2 of $Q_{c,r}$ let $\Delta(x^1, x^2) = (M^1 \setminus M^2) \cup (M^2 \setminus M^1)$. Since M^1, M^2 are both feasible perfect matchings in the same bipartite graph, it follows that $\Delta(x^1, x^2)$ is a collection of mutually disjoint even simple cycles in $K_{n,n}$, each cycle consisting alternately of an edge in M^1 and an edge in M^2 . If p is the number of such cycles, then $\Delta(x^1, x^2) = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \dots \cup \mathcal{C}_p$ where \mathcal{C}_t is the t th simple cycle in $\Delta(x^1, x^2)$ for $t = 1$ to p . If $p = 1$, x^1, x^2 are adjacent on K_A itself, and hence are also adjacent on $Q_{c,r}$. If $p > 1$, define for $t = 1$ to p , the cost of \mathcal{C}_t to be $\alpha_t = \sum_{(i,j) \in M^1 \cap \mathcal{C}_t} c_{ij} - \sum_{(i,j) \in M^2 \cap \mathcal{C}_t} c_{ij}$. Since both x^1 and $x^2 \in Q_{c,r}$, we have $\sum_{(i,j) \in M^1} c_{ij} = \sum_{(i,j) \in M^2} c_{ij} = r$, this implies that $\sum_{t=1}^p \alpha_t = 0$.

Lemma 1. Let x^1, x^2 be two distinct feasible assignments. x^1, x^2 are nonadjacent on $Q_{c,r}$ iff there exist two distinct feasible assignments x^3, x^4 , both distinct from x^1, x^2 satisfying $\frac{1}{2}(x^1 + x^2) = \frac{1}{2}(x^3 + x^4)$ and $\Delta(x^1, x^2) = \Delta(x^3, x^4)$.

Proof. The “if” part is obvious, since if such x^3, x^4 exist, by definition x^1, x^2 are nonadjacent on $Q_{c,r}$.

Now to prove the “only if” part, suppose x^1, x^2 are nonadjacent on $Q_{c,r}$. Then by definition [13], there exist distinct extreme points y^1, \dots, y^u of $Q_{c,r}$, each of them distinct from x^1, x^2 , such that the following system is feasible in $\beta_1, \beta_2, \gamma_1, \dots, \gamma_u$.

$$\begin{aligned} \beta_1 x^1 + \beta_2 x^2 &= \sum_{v=1}^u \gamma_v y^v, \\ \beta_1 + \beta_2 &= \sum_{v=1}^u \gamma_v = 1, \end{aligned} \quad (2)$$

$$\beta_1, \beta_2, \gamma_1, \dots, \gamma_u > 0.$$

Let M^1, M^2 be the feasible perfect matchings corresponding to x^1, x^2 , and N^v the feasible perfect matching corresponding to y^v for $v = 1$ to u . From (2) we conclude that $M^1 \cup M^2 = \bigcup_{v=1}^u N^v$, and that if $E = M^1 \cap M^2 \neq \emptyset$, then $E \subset N^v$ for all $v = 1$ to u . Define

$$\bar{N}^1 = E \cup [(M^1 \setminus M^2) \cup (M^2 \setminus M^1) \setminus (N^1 \setminus E)].$$

It can be verified that the above facts imply that \bar{N}^1 is also a perfect matching, and we have $M^1 \cup M^2 = N^1 \cup \bar{N}^1$ and $M^1 \cap M^2 = N^1 \cap \bar{N}^1 = E$. Also, since x^1, x^2, y^1 are all feasible assignments, we have $\sum_{(i,j) \in M^1} c_{ij} = \sum_{(i,j) \in M^2} c_{ij} = \sum_{(i,j) \in N^1} c_{ij} = r$, and hence, $\sum_{(i,j) \in \bar{N}^1} c_{ij} = \sum_{(i,j) \in M^1 \cup M^2} c_{ij} - \sum_{(i,j) \in N^1} c_{ij} = r$, so; \bar{y}^1 , the assignment corresponding to the perfect matching \bar{N}^1 , is also feasible.

So, if we define $x^3 = y^1$ and $x^4 = \bar{y}^1$, then both x^3, x^4 are distinct feasible assignments both distinct from x^1, x^2 , and satisfy $\frac{1}{2}(x^1 + x^2) = \frac{1}{2}(x^3 + x^4)$ and $\Delta(x^1, x^2) = \Delta(x^3, x^4)$, proving the lemma. \square

Theorem 1. Let x^1, x^2 be two distinct feasible assignments. Let $\Delta(x^1, x^2)$ consist of p cycles $\mathcal{C}_1, \dots, \mathcal{C}_p$, and let their costs defined as above be $\alpha_1, \dots, \alpha_p$. Then x^1, x^2 are

nonadjacent on $Q_{c,r}$ iff there exists a proper subset of $\{\alpha_1, \dots, \alpha_p\}$ whose sum is zero. In particular, x^1, x^2 are adjacent on $Q_{c,r}$ if $p = 1$.

Proof. When $p = 1$, from the well-known results on the assignment polytope K_A [2, 6], x^1, x^2 are adjacent on K_A and consequently on $Q_{c,r}$.

Consider the case $p > 1$. Suppose that x^1, x^2 are not adjacent on $Q_{c,r}$. From Lemma 1 there must exist two feasible assignments x^3, x^4 in $Q_{c,r}$ both distinct from x^1, x^2 satisfying $\frac{1}{2}(x^1 + x^2) = \frac{1}{2}(x^3 + x^4)$ and $\Delta(x^1, x^2) = \Delta(x^3, x^4)$.

Let M^t be the feasible perfect matching corresponding to x^t for $t = 1$ to 4. Then $M^1 \cap M^2 = M^3 \cap M^4 = E$ say, and since x^1 is distinct from both x^3 and x^4 , it follows that both $(M^1 \cap M^3) \setminus E$ and $M^3 \setminus M^1$ are nonempty. Thus, $\Delta(x^1, x^3)$ is a proper subset of $\Delta(x^1, x^2)$ consisting of a collection of simple cycles which is a proper subset of $\{\mathcal{C}_1, \dots, \mathcal{C}_p\}$. Again, since $x^1, x^3 \in Q_{c,r}$, we have $\sum_{(i,j) \in M^3} c_{ij} = \sum_{(i,j) \in M^1} c_{ij} = r$, this implies that $\sum(\alpha_k : k \text{ such that } \mathcal{C}_k \text{ is in } \Delta(x^1, x^3)) = 0$. Hence in this case $\{\alpha_k : k \text{ such that } \mathcal{C}_k \text{ is in } \Delta(x^1, x^3)\}$ is a proper subset of $\{\alpha_1, \dots, \alpha_p\}$ whose sum is zero.

To prove the converse, suppose a proper subset $\{\alpha_{k_1}, \dots, \alpha_{k_s}\}$ satisfies the property that $\alpha_{k_1} + \dots + \alpha_{k_s} = 0$. Define

$$\begin{aligned} L_1 &= \{(i, j) : (i, j) \text{ contained on the simple cycles in the set } \{\mathcal{C}_{k_1}, \dots, \mathcal{C}_{k_s}\}\} \\ L_2 &= \{(i, j) : (i, j) \text{ contained on the simple cycles in the set } \{\mathcal{C}_1, \dots, \mathcal{C}_p\} \setminus \\ &\quad \{\mathcal{C}_{k_1}, \dots, \mathcal{C}_{k_s}\}\}. \end{aligned}$$

Define x^3 and x^4 by

$$\begin{aligned} M^3 &= (M^1 \cap M^2) \cup (M^1 \cap L_1) \cup (M^2 \cap L_2), \\ M^4 &= (M^1 \cap M^2) \cup (M^2 \cap L_1) \cup (M^1 \cap L_2). \end{aligned}$$

It can be verified that M^3, M^4 are perfect matchings on G . The fact that $\alpha_{k_1} + \dots + \alpha_{k_s} = 0$ implies that $\sum_{(i,j) \in M^1 \cap L_1} c_{ij} = \sum_{(i,j) \in M^2 \cap L_1} c_{ij}$, and since $M^3 \cup M^4 = M^1 \cup M^2$ it follows that $x^3, x^4 \in Q_{c,r}$. Therefore, x^1, x^2 are not adjacent on $Q_{c,r}$ since $\frac{1}{2}(x^1 + x^2) = \frac{1}{2}(x^3 + x^4)$. \square

Theorem 2. Checking whether two given extreme points of $Q_{c,r}$ are nonadjacent on it is NP-complete if $c = (c_{ij})$ is a general integer matrix; and is solvable with at most $\mathcal{O}(n^5)$ effort if c is a 0–1 matrix.

Proof. Given feasible assignments x^1, x^2 , we find the cycles in $\Delta(x^1, x^2)$ and the set of their costs $\{\alpha_1, \dots, \alpha_p\}$. When c is a general integer matrix, $\alpha_1, \dots, \alpha_p$ are general integers. By Theorem 1, x^1 and x^2 are nonadjacent on $Q_{c,r}$ iff there exists a proper subset of $\{\alpha_1, \dots, \alpha_p\}$ whose sum is zero. Checking this is NP-complete [7] (since the numbers $\alpha_1, \dots, \alpha_p$ can be given arbitrary integer values by the use of an appropriate choice of the costs c). Hence, checking nonadjacency of x^1, x^2 on $Q_{c,r}$ is NP-complete in this case.

If c is a 0–1 matrix, all of $\alpha_1, \dots, \alpha_p$, are integers between $-n$ and $+n$. When all of $\alpha_k, \dots, \alpha_p$ satisfy these bounds, checking the existence of a proper subset of $\{\alpha_1, \dots, \alpha_p\}$

whose sum is zero can be carried out in $\mathcal{O}(n^5)$ time via *dynamic programming*, see the algorithm given below. \square

Algorithm to check nonadjacency of feasible assignments x^1, x^2 on $Q_{c,r}$.

Begin

- 1 Find the simple cycles $\{\mathcal{C}_1, \dots, \mathcal{C}_p\}$ in $\Delta(x^1, x^2)$. If $p = 1$ go to 3. If $p > 1$, let $\alpha_1, \dots, \alpha_p$ be the costs of the simple cycles $\mathcal{C}_1, \dots, \mathcal{C}_p$ as defined above. If any of $\alpha_1, \dots, \alpha_p$ are zero, go to 4; otherwise go to 2.
- 2 All of $\alpha_1, \dots, \alpha_p$ are nonzero. Let $\gamma = -1 + \sum_{\alpha_i > 0} \alpha_i$. For $\beta = 1$ to γ do:
 - 2.1 Solve $\sum_{\alpha_i > 0} \alpha_i x_i = \beta$, $x_i = 0$ or 1 for all i .
If this system has a solution, go to 2.2. Otherwise go to the next β if $\beta < \gamma$, or to 3 if $\beta = \gamma$.
 - 2.2 Solve $\sum_{\alpha_i < 0} |\alpha_i| y_i = \beta$, $y_i = 0$ or 1 for all i .
If this system has a solution, go to 4. Otherwise go to the next β if $\beta < \gamma$, or to 3 if $\beta = \gamma$.
- 3 There exists no proper subset of $\{\alpha_1, \dots, \alpha_p\}$ whose sum is 0, so x^1, x^2 are adjacent on $Q_{c,r}$, terminate.
- 4 There exists a proper subset of $\{\alpha_1, \dots, \alpha_p\}$ whose sum is 0, so x^1, x^2 are non-adjacent on $Q_{c,r}$, terminate.

End

Steps 2.1 and 2.2 can be carried out by dynamic programming in $\mathcal{O}(n\beta)$ [15]. Since $\gamma \leq n^2$, the overall complexity of this algorithm is $\mathcal{O}(n^5)$.

Remark. Let $r = n$, $c_{ij} = 1$ for all $i, j = 1$ to n . Then, $Q_{c,n} = K_A$ since the side constraint $\sum_i \sum_j c_{ij} x_{ij} = r$ is redundant. In this case, $\alpha_t = 0$ for all $t = 1$ to p . Thus, the problem of checking whether x^1, x^2 are adjacent on $Q_{c,n}$ reduces to the problem of determining whether $\Delta(x^1, x^2)$ is connected. This reduces to Chvátal's adjacency result on the assignment polytope [6].

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